XXIII. *An essay towards the calculus of functions.* *By* C. Babbage, *Esq.* *Communicated by W. H. Wollaston, M. D. Sec. R. S.*

Read June 15, 1815.

THE term function has long been introduced into analysis with great advantage, for the purpose of designating the result of every operation that can be performed on quantity. This extent of signification has rendered it of essential use, but the various applications of which it admits, and the questions to which it gives rise, do not appear to have met with sufficient attention.

I propose in the following paper to present an outline of a new calculus, which naturally results from it. It comprehends questions of the greatest generality and difficulty, and will probably require the invention of new methods for its improvement.

Many of the calculations with which we are familiar, consist of two parts, a direct, and an inverse; thus, when we consider an exponent of a quantity: to raise any number to a given power, is the direct operation: to extract a given root of any number, is the inverse method. The differential calculus, which is a direct method, naturally gave rise to the integral, which is its inverse: the same remark is applicable to finite differences. In all these cases the inverse method is by far the most difficult, and it might perhaps be added, the most useful.

It is this inverse method with respect to functions, which I at present propose to consider.

If an unknown quantity as x, be given by means of an equation, it becomes a question how to determine its value; similarly if an unknown function as ψ, be given by means of any functional equation, it is required to assign its form. In the first case, it is quantity which is to be determined; in the second, it is the form assumed by quantity, that becomes the subject of investigation. In the one case, the various powers of the unknown quantity enter into the equation; in the other, the different orders of the function are concerned.

Before I proceed, it will be proper to explain the meaning of the order of a functional equation, and likewise to indicate the notation made use of; α, β, γ, &c. are known functional characteristics; ψ, χ, ψ, are unknown ones.

If in any function as ψx, instead of x, the original function be substituted, it becomes $\psi\psi x$ or $\psi^2 x$: this is called the second function of x. If the process be repeated, the result is $\psi^2\psi x$ or $\psi^3 x$, the third function of x; and similarly $\psi^n x$, denotes the n^{th} function of x. Suppose

$$\psi x = a + x$$

then $$\psi^2 x = a + a + x = 2a + x$$

and generally $$\psi^n x = na + x$$

A functional equation is said to be of the first order, when it contains only the first function of the unknown quantity; as, for instance,

$$\psi\alpha x + x\psi x - x^n = o$$

$$\left(\psi x + \psi\frac{1}{x}\right)^n - ax + x^2 = o.$$

If the second function enter, the equation rises to the second order: thus,

$$\psi^2 x = x$$

$$\psi(x + \psi x) + (\psi x - x)^2 = o$$

$$\left(\psi^2 x + \psi \frac{1}{x}\right)^n = ax$$

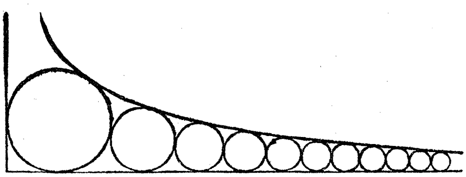
A function of two variables admits of two second functions: thus $\psi(x,y)$ becomes $\psi(\psi(x,y),y)$, and $\psi(x,\psi(x,y))$ or they might be thus expressed $\psi^{2,1}(x,y)$ and $\psi^{1,2}(x,y)$.

These express the second functions; the first taken relative to x, the other relative to y. But besides these two there is another, which arises from taking the second function simultaneously relative to x, and y; it is $\psi\left\{(\psi x,y),\psi(x,y)\right\}$. This ought not to be written $\psi^{2,2}(x,y)$ for it is not the second function first taken relative to x and then to y, nor is it the converse of this. In fact, the notation is defective; some method is wanting of indicating the order in which the successive substitutions are made. I shall for the present lay aside the consideration of functional equations, involving more than one variable.

Those of the first order have long been known, but the method in which I have treated them is, I believe, entirely new. Equations of the second and higher orders have never been even mentioned; it is these which present the most interesting speculations, and which are involved in the greatest difficulties. I shall first give some account of the enquiries which led me to this subject, and shall then treat of the various orders of functional equations.

Some few years since, while considering a problem mentioned by PAPPUS, relating to the inscription of a number

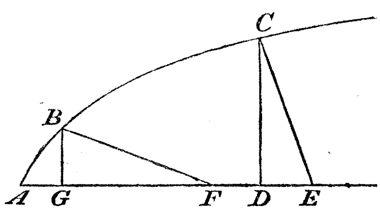
of circles in a semicircle, the following question occurred to me.



If in an hyperbola between its assymptotes a circle be inscribed touching both asymptotes and the curve, and if another circle be inscribed touching the first circle, the curve, and one asymptote, and if this be continued as represented in the figure, what ratio does the area of the circles bear to that of the figure; and conversely, if this ratio is given, what is the nature of the curve? I soon perceived the great difficulty of the subject, and that these and other problems similar to the latter of them, required the application of methods totally different from any with which I was then acquainted.

Hopeless of success, I laid aside the subject until about two years after, when the same difficulty recurred under another form. I had proposed to myself the following problem:

What must be the nature of the curve ABC, such that if



any point C be taken, and the ordinate CD, the normal CE, and subnormal DE be drawn, and if the triangle CDE be turned into such a position that CD may become the base and DE the perpendicular, if DE coincide with some new ordinate as GB, then the normal CE at the first point shall coincide with the normal BF at the second?

This latter question, of far less difficulty than the former, was readily solved, but I did not at first observe that they rested on similar principles; this, however, was pointed out by Mr. HERSCHEL, to whom I had mentioned the subject. Such was the origin of the following enquiries.

The question, in its most general point of view, is the solution of functional equations of all orders.

This, however, is a generality which I do not pretend to have attained. In the first part of this Paper the reader will find a new method of solving all functional equations of the first order; it depends on possessing their particular solutions. In the subsequent part, I have given various methods of solving functional equations of the second and higher orders: some of these possess considerable generality; and if we consider only those in which the n^{th} and inferior orders enter simply, such as

$$F\left\{x, \psi x, \psi^2 x, \ldots \psi^n x\right\} = o$$

I have pointed out the means of obtaining their solutions.

The determination of functions from given conditions most probably took its rise from the integration of equations of partial differentials; and we accordingly find that the authors of this calculus were soon engaged in the new problem to which it gave birth. D'ALEMBERT was the first who occupied himself with this subject: he was soon followed by EULER and LAGRANGE; but it is to MONGE that we are indebted for the most general view of the subject. His enquiries were directed to the determination of two functions from given conditions; they are contained in the fifth volume of the *Melanges de Turin*, and in two Papers in the seventh volume of the *Memoires des Savans Etrangers*, 1773.

In the first of these he explains the solution of several functional equations by means of curves of double curvature, and by curve surfaces.

In the second Paper, the question is treated in a more analytical method, and he endeavours to reduce it to the solution of equations of differences. "Je me propose," observes MONGE, "de faire voir que la determination des fonctions arbitraires qui se trouvent dans l' integrale d'une equation aux différences partielles, depend en general, dans les cas que je n'ai pas encore traités, de l'integrale d'une ou de plusieurs équations aux differences finies, dans lesquelles le rapport de la variable principale à sa difference finie est donné soit qu'il soit variable soit qu'il soit constant."

In the same volume is a paper of LAPLACE on this subject, which he views in the same light, and endeavours to reduce functional equations of the first order to those of finite differences. This skilful analyst first solved the functional equation $F\left\{x, \phi x, \phi\alpha x,\right\} = o$. The method he made use of is peculiarly elegant; he converted it into an equation of finite differences in which the difference was constant. Still, however, it appeared by no means the most direct method to make use of such an expedient, nor was it even known that all equations of the first order admitted of its application. This latter objection was, however, removed by Mr. HERSCHEL, who in an excellent paper on functional equations, has extended the method made use of by LAPLACE to the solution of all equations of the first order. His solution is equally elegant and general; it leaves nothing to be regretted, but the narrow limits of our knowledge respecting the integration of equations of finite differences. From this and other causes, I am still

inclined to think that the solution of functional equations must be sought by methods peculiarly their own. There are some other researches on this difficult subject of which I am unable to give any account, from the impossibility of procuring the works in which they are contained; among these is the paper of ARBOGAST, which gained the prize of the Academy of Petersburg in the year 1790.

For the sake of convenience, I shall call any solution of a functional equation which contains one or more arbitrary functions, a general solution; but if the solution of such an equation only contains arbitrary constants, I shall call it a particular solution. With respect to the number of arbitrary functions that may enter into any solution, I shall make some observations at the conclusion of this paper.

PROBLEM I.

Required the general solution of the functional equation,

$$\psi x = \psi \alpha x$$

supposing we are acquainted with one particular solution.

Let the particular solution be $fx = f\alpha x$; then take $\psi = \varphi f$, φ being an arbitrary function. It is evident that this value of ψ will satisfy the original equation, and that

$$\varphi f x = \varphi f \alpha x$$

is identical, because $fx = f\alpha x$.

Example, let the equation be

$$\psi(x) = \psi(-x)$$

and the particular solution be

$$fx = x^2$$

the general solution is

$$\psi x = \varphi(x^2)$$

which evidently answers the conditions.

As I shall have frequent occasion to make use of symmetrical functions of two or more quantities, I shall for the sake of brevity denote this by putting a line over the functional characteristic; thus $\bar{\varphi}\,(x,y)$ represents a symmetrical function of x and y, which it is well known possesses the following property,

$$\bar{\varphi}\left\{x,y\right\} = \bar{\varphi}\left\{y,x\right\}$$

As we are only considering functional equations of one variable, this will be sufficient for the present purpose; it might perhaps otherwise be more advisable to put the line over the quantities relative to which the function is symmetrical; thus $\varphi\left\{x, y, \bar{z}, \bar{v}\right\}$ is symmetrical relative to z and v, but it is not so in respect to the other variables. This would possess the advantage of readily designating a function symmetrical relative to two quantities in one way, and likewise symmetrical with respect to two others, but in a different manner,* thus

$$\varphi\left\{\bar{x}, \bar{y}, \overset{1}{\bar{v}}, \overset{1}{\bar{z}}\right\}$$

a particular case of this is

$$\frac{v+z+axy}{v^3\,z^3 - a\,x^2\,y^2}$$

which is symmetrical in one sense relative to x and y, and in a different sense with respect to v and z; but these belong to other enquiries.

PROBLEM II.

Required a general solution of the equation $\psi\,x = \psi\,\alpha\,x$, having given a particular solution of $f\,x = f\,\alpha^2\,x$

* This is not a mere imaginary refinement; I have constantly had occasion to make use of functions of many variables which were symmetrical by pairs, when investigating the nature of functional equations of more than one variable.

Assume $\psi x = \bar{\varphi}\{f x, f_1 x\}$ then it becomes

$$\bar{\varphi}\{f x, f_1 x\} = \bar{\varphi}\{f \alpha x, f_1 \alpha x\}$$

this equation will be satisfied if we determine f and f_1 so that the following equations may be fulfilled.

$$f \alpha x = f_1 x \text{ and } f_1 \alpha x = f x$$

from these result $f \alpha^2 x = f_1 \alpha x = f x$.

but we have by hypothesis a particular solution of $f \alpha^2 x = f x$,
therefore the general solution of $\psi x = \psi \alpha x$ is

$$\psi x = \bar{\varphi}\{f x, f \alpha x\}$$

If the function α should be of such a nature that $\alpha^2 x = x$, or even if $\alpha^{2n} x = x$, $f x$ will then become x.
for example, suppose $\psi x = \psi \left(\frac{x}{ax-1}\right)$

then since $\alpha x = \frac{x}{ax-1}$, $\alpha^2 x = \alpha \left(\frac{x}{ax-1}\right) = \dfrac{\frac{x}{ax-1}}{a\frac{x}{ax-1} - 1} = x$

and we have $\psi x = \bar{\varphi}\left\{x, \frac{x}{ax-1}\right\}$
a particular case of this is when $a = o$, $\psi(x) = \psi(-x)$, its solution is

$$\psi x = \bar{\varphi}\{x, -x\} = \varphi(x^2)$$

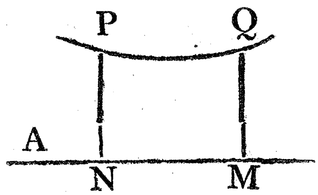
the same as in the last Problem; as another example take

$$\psi x = \psi \frac{1}{x}$$

then since $\alpha x = \frac{1}{x}$, $\alpha^2 x = \dfrac{1}{\frac{1}{x}} = x$ and $\psi x = \bar{\varphi}\left\{x, \frac{1}{x}\right\}$.

This affords a solution to the following question:

Required the nature of a curve, such that if any two abscissæ, whose rectangle is equal to a given square be taken, their corresponding ordinates may also be equal.



Let $y = \psi\, x$ be the equation of the curve, also $\mathrm{AN} = x$, and if $\mathrm{AN} \times \mathrm{AM} = a^2$, then the property of the curve is that

$$\mathrm{PN} = \mathrm{QM},$$

but $\mathrm{PN} = y = \psi x$, and $\mathrm{QM} = \psi\, (\mathrm{AM}) = \psi \left(\frac{a^2}{x}\right)$ consequently the equation from which ψ must be determined, is

$$\psi x = \psi \frac{a^2}{x}$$

its solutios in

$$y = \psi\, x = \bar{\varphi}\left\{x, \frac{a^2}{x^2}\right\},$$

if we make $y = x + \frac{a^2}{x}$, it becomes $yx - x^2 = a^2$, an equation to the hyperbola.

Problem III.

Required the general solution of $\psi\, x = \mathrm{A}\, x \times \psi\, \alpha\, x$; having a particular solution, and also one of $\psi x = \psi\, \alpha^2\, x$.

Assume $\psi\, x = fx \times \bar{\varphi}\, \left\{\underset{1}{f}x, \underset{2}{f}x\right\}$

making this substitution in the original equation, it becomes $fx \times \bar{\varphi}\, \left\{\underset{1}{f}x, \underset{2}{f}x\right\} = \mathrm{A}\, x \times f\alpha\, x\, \bar{\varphi}\, \left\{\underset{1}{f}\alpha x, \underset{2}{f}\alpha x\right\}$ make $\underset{2}{f}\, \alpha\, x = \underset{1}{f}x$, and $\underset{1}{f}\alpha x = \underset{2}{f}x$, from this it results that $\underset{1}{f}x = \underset{1}{f}\alpha^2\, x$ of which we possess a particular solution, divide both sides by $\bar{\varphi}\left\{\underset{1}{f}x, \underset{1}{f}\alpha x\right\}$ then we have

$$fx = \mathrm{A}x \times f\, \alpha x\,;$$

this is nothing more than the original equation of which, and of $\underset{1}{f}x = \underset{1}{f}\alpha^2\, x$, we have by hypothesis particular solutions:

therefore its general solution is,

$$\psi x = fx \times \bar{\varphi}\{\underset{1}{f}x, \underset{1}{f}\alpha x\}$$

The same equation may be solved differently, if we are acquainted with particular solutions of the equations

$$\psi x = \psi \alpha x, \text{ and } \psi x = \mathrm{A} x \times \psi \alpha x.$$

Assume $\psi x = fx \times \varphi x$, it becomes

$$fx \times \varphi x = \mathrm{A} x \times f \alpha x \times \varphi \alpha x,$$

let fx be the particular solution of the original equation, then

$$fx = \mathrm{A} x \times f \alpha x,$$

and by division

$$\varphi x = \varphi \alpha x,$$

but of this also we have given a particular solution, call it $\underset{1}{f}x$, therefore the general expression of ψx, is

$$\psi x = fx \times \varphi \underset{1}{f} x.$$

Let us take for example the equation

$$\psi x = \frac{1-x^2}{1-2x} \psi \frac{x}{x-1}$$

the particular value of ψx is

$$fx = 1 + x,$$

and the particular case of

$$\psi x = \psi \frac{x}{2x-1} \qquad \text{is} \qquad \underset{1}{f} x = \frac{x^2}{x-1},$$

from these considerations the general solution is

$$\psi x = (1+x)\, \varphi \frac{x^2}{x-1}$$

which will on trial be found to satisfy the equation.*

* Another particular solution of

$$\psi x = \psi \frac{x}{x-1} \text{ is } \psi x = \cos. \log. (x-1)$$

this gives for the general solution of

$$\psi x = \frac{1-x^2}{1-2x} \psi \frac{x}{x-1}$$

$$\psi x = (1+x)\, \varphi\, (\cos. \log. (x-1)).$$

As another example, take

$$\psi x = -\frac{1}{x}\psi\frac{1}{x}$$

the particular solutions are, that of the equation itself

$$f x = \frac{x-1}{x}$$

and that of $\psi x = \psi\frac{1}{x}$ is known to be $\bar{\varphi}\left(x, \frac{1}{x}\right)$, hence the general solution is

$$\psi x = \frac{x-1}{x}\bar{\varphi}\left(x, \frac{1}{x}\right)$$

which, as is readily seen, fulfills the condition.

PROBLEM IV.

Required the general solution of

$$\psi x = \mathrm{A}x \times \psi \alpha x + \mathrm{B}x,$$

having given a particular solution fx

Assume $\psi x = fx + \varphi x$, then $\psi \alpha x$ becomes $f\alpha x + \varphi \alpha x$, and the equation is

$$fx + \varphi x = \mathrm{A}x \times f\alpha x + \mathrm{A}x \times \varphi \alpha x + \mathrm{B}x,$$

subtracting from this the particular solution $fx = \mathrm{A}x \times f\alpha x + \mathrm{B}x$, which is given by hypothesis, there remains

$$\varphi x = \mathrm{A}x \times \varphi \alpha x,$$

this equation may be solved by Prob. III, and we thence obtain the general solution of the given equation.

From this may readily be deduced the solution of the following equations:

$$\psi(\varepsilon^y) \pm (\varepsilon^{-y}) = \mathrm{A}y$$

suppose f is a particular solution, take $\psi y = fy + \chi y$, the equation becomes

$$f(\varepsilon^y) \pm f(\varepsilon^{-y}) + \chi(\varepsilon^y) \pm \chi(\varepsilon^{-y}) = \mathrm{A}y,$$

but we have from the particular solution

$$f(\varepsilon^y) \pm f(\varepsilon^{-y}) = \mathrm{A}y,$$

this subtracted from the former, leaves

$$\chi(\varepsilon^{y}) \pm \chi(\varepsilon^{-y}) = o.$$

Let us first consider the upper sign, then a particular solution of

$$\chi(\varepsilon^{y}) = -\chi(\varepsilon^{-y}),$$

$$\text{is } \chi(y) = (\log. y)^{2n+1}.$$

If we take the lower sign, then a solution of

$$\chi(\varepsilon^{y}) = \chi(\varepsilon^{-y})$$

$$\text{is } \chi(y) = (\log. y)^{2n}.$$

From these considerations it appears, that the general solutions of the given equations are

$$\psi y = fy + \varphi\left\{(\log. y)^{2n+1}\right\}$$

$$\text{and } \psi y = fy + \varphi\left\{(\log. y)^{2n}\right\}$$

according as the upper or under sign is used.

The equation just solved was not constructed as an example to this particular rule, but is selected because it has actually occurred. It is used by Mr. HERSCHEL in the Philosophical Transactions, 1814, for the purpose of assigning the sums of several very curious series. He there observes, that when the upper sign is used ${}^{2n}L(1+y)$, and when the lower takes place ${}^{2n+1}L(1+y$, are particular solutions, these may therefore be generalized by the introduction of an arbitrary function

If $\psi\varepsilon^{y} + \psi\varepsilon^{-y}$ = a rational function of y containing only even powers,

or if $\psi\varepsilon^{y} - \psi\varepsilon^{-y}$ = a rational function of y containing only odd powers,

they admit of the following solutions, in the first case, let

$$\psi\varepsilon^{y} + \psi\varepsilon^{-y} = \underset{0}{a} + \underset{1}{a}y^{2} + \underset{2}{a}y^{4} + \&c. + \underset{n}{a}y^{2n}$$

assume $\psi y = \underset{0}{A} + \underset{1}{A} (\log. y)^2 + \&c. + \underset{n}{A} (\log. y)^{2n}$

$\psi \varepsilon^y$ becomes $\underset{0}{A} + \underset{1}{A} y^2 + \underset{2}{A} y^4 + \&c. + \underset{n}{A} y^{2n}$

$\psi \varepsilon^{-y}$ becomes $\underset{0}{A} + \underset{1}{A} y^2 + \underset{2}{A} y^4 + \&c. + \underset{n}{A} y^{2n}$

By comparing the co-efficients,

$$\underset{0}{A} = \tfrac{1}{2} \underset{0}{a} \quad \underset{1}{A} = \tfrac{1}{2} \underset{1}{a}, \quad \&c. \quad \underset{n}{A} = \tfrac{1}{2} \underset{n}{a}$$

and calling the right side of the equation $F(y^2)$, if

$$\psi \varepsilon^y + \psi \varepsilon^{-y} = F y^2,$$

a particular solution is

$$\psi y = \tfrac{1}{2} F \left\{ (\log. y)^2 \right\}$$

and similarly if

$\psi \varepsilon^y - \psi \varepsilon^{-y} = F(y) =$ a function containing only odd powers of y, one particular solution is

$$\psi y = \tfrac{1}{2} F \left\{ \log. y \right\}$$

and the general solutions may be readily deduced as above.

PROBLEM V.

To reduce the equation $\psi x + A x \times \psi \alpha x + \&c. + N x \times \psi \nu x + X = o$, to one in which the last term is wanting, by means of a particular solution,

Let $f x$ be the given solution, make $\psi x = f x + \varphi x$, and substituting this value, the equation becomes

$$\left. \begin{array}{l} f x + A x \times f \alpha x + B x \times f \beta x + \&c. + N x \times f \nu x + X \\ + \varphi x + A x \times \varphi \alpha x + B x \times \varphi \beta x + \&c. + N x \times \varphi \nu x \end{array} \right\} = o$$

the upper line is by hypothesis equal to nothing, therefore the equation is reduced to this,

$$\varphi x + A x \times \varphi \alpha x + \&c. + N x \times \varphi \nu x = o \qquad (1)$$

and if we can discover the general solution of this latter equation, that of the former may be readily found.

Supposing α, β, γ, &c. ν, were to become $\alpha, \alpha^2, \alpha^3, \ldots \alpha^n$, the equation (1) would be changed into

$\phi x + \mathrm{A}x \times \phi\alpha x + \mathrm{B}x \times \phi\alpha^2 x + \&c. + \mathrm{N}x \times \phi\alpha^{n-1} x = o$ (2)

after further if αx, is such a function of x, that $\alpha^n x = x$, if we are acquainted with one particular solution of (2), we may easily determine the general one thus:

Assume $\phi x = fx \times \bar{\chi}\left\{x, \alpha x, \alpha^2 x, \ldots \alpha^{n-1} x\right\}$

here we must observe, that since χ is symmetrical relative to all the quantities contained within the brackets, it is immaterial in what order they are placed, and from the condition that $\alpha^n x = x$, it follows that if we substitute $\alpha^k x$ for x, (k being successively equal to 1, 2, 3, 4, and $n-1$, we shall always have these values $x, \alpha x, \alpha^2 x \ldots \alpha^{n-1}x$ only differently arranged, from these considerations the equation (2) will become

$o = (fx + \mathrm{A}x \times f\alpha x + \mathrm{B}x \times f\alpha^2 x + \&c. + \mathrm{N}x \times f\alpha^{n-1} x)$
$\bar{\chi}\left\{x, \alpha x, \alpha^2 x, \ldots \alpha^{n-1} x\right\}$;

this equation may be satisfied by making the factor which multiplies $\bar{\chi}$ equal to nothing, and this is always the case when f is a particular solution, hence

$$\phi x = fx \times \bar{\chi}\left\{x, \alpha x, \alpha^2 x, \ldots \alpha^{n-1} x\right\}.$$

Problem VI.

To find a function of x, such that if instead of x we successively substitute αx, βx, γx, &c. νx, the results shall all be equal to the original function; or in other words, to determine ϕx from the equations

$$\phi x = \phi\alpha x = \phi\beta x = \&c. = \phi\nu x$$

find f, so as to satisfy the equation

$$f x = f \alpha x \qquad (a)$$

take any particular value

Find f_1, so as to satisfy the equation

$$f_1 f x = f_1 f \alpha \beta x = f_1 f \beta x \qquad (b)$$

for it is known from (a) that

$$f = f \alpha$$

take some particular value, and determine f_2 from the equation

$$f_2 f_1 f x = f_2 f_1 f \beta \gamma x = f_2 f_1 f \gamma x,$$

continue this as far as

$$f_n f_{n-1} \ldots f x = f_n f_{n-1} \ldots f \nu x \qquad (n)$$

then φ being any arbitrary function

$$\varphi \left\{ f_n f_{n-1} \ldots f_1 f x \right\}$$

will satisfy the conditions of the problem.

As an example, let it be required to find a function which shall not change, when for x we substitute x, $-x$, or $\frac{x}{\sqrt{x^2-1}}$ here $\alpha x = - x$, and $\beta x = \frac{x}{\sqrt{x^2-1}}$

(a) becomes $f(x) = f(-x)$,

hence $f x = \bar{\varphi} \left\{ x, -x \right\}$ as a particular case, take $f x = x^2$, then (b) becomes

$$f_1 x^2 = f_1 (\beta x)^2 = f_1 \frac{x^2}{x^2-1}$$

whose solution is $f_1 x = \bar{\varphi} \left\{ x, \frac{x}{x-1} \right\}$

take the case of $f_1 x = \frac{x^2}{x-1}$

then we find

$$\phi f_1 f x = \phi\left\{\frac{x^4}{x^2-1}\right\}$$

which fulfils the given conditions.

In the same manner it may be found, that the function

$$\phi\left\{\frac{1-ax^2+x^4}{1+ax^2+x^4}\right\}$$

will remain the same, whether the variable is x, $-x$, or $\frac{1}{x}$.

Problem VII.

Given any two series of functions αx, βx, &c. νx and $\alpha_1 x$, $\beta_1 x$, &c. $\nu_1 x$, required the form of ψ, so that the following equations may be fulfilled,

$$\begin{aligned} \psi\alpha x &= \psi\alpha_1 x \\ \psi\beta x &= \psi\beta_1 x \\ \text{\&c.}\quad & \quad \text{\&c.} \\ \psi\nu x &= \psi\nu_1 x. \end{aligned}$$

Determine f from the equation

$$f\alpha x = f\alpha_1 x \qquad (a)$$

take some particular case, and determine f_1 from the equation

$$f_1 f\alpha\beta x = f_1 f\alpha_1\beta_1 x = f_1 f\alpha\beta_1 x \qquad (b)$$

take a particular case and find f_2 from the equation

$$f_2 f_1 f\alpha\beta\gamma x = f_2 f_1 f\alpha\beta_1\gamma_1 x = f_2 f_1 f\alpha\beta\gamma_1 x \qquad (c)$$

and continue this to

$$f_n f_{n-1}\dots f\alpha\beta\gamma\dots\nu x = f_n f_{n-1}\dots f\alpha\beta\gamma\dots\nu_1 x \qquad (n)$$

Then will

$$\psi x = \phi \underset{n}{f} \underset{n-1}{f} \ldots f \alpha \beta \gamma \ldots \nu x$$

satisfy the conditions.

We now possess the means of solving a much more general problem than we have yet attempted, it is the general solution of any functional equation of the first order.

PROBLEM VIII.

Required the general solution of the equation

$$F \left\{ x, \psi x, \psi \alpha x, \ldots \psi \nu x \right\}$$

having given a particular one, containing one or more arbitrary constants.

Find from Problem VI. an arbitrary function, such that it shall remain the same, when αx, βx, &c. νx, are substituted for x, call this ϕx, and let the particular solution be

$$\psi x = f \left\{ x, a, b, c, \text{\&c.} \right\}$$

As these arbitrary constants are supposed not to be contained in the given equation $F \left\{ x, \psi x, \psi \alpha x, \ldots \psi \nu x \right\} = o$, when the particular solution is substituted, they must destroy each other, whatever be their value.

If therefore, instead of a, b, c, &c. we put ϕx, $\underset{1}{\phi} x$, $\underset{2}{\phi} x$, &c. (which are arbitrary functions fulfilling this condition, viz. that they remain identical when αx, βx, &c. are put for x:) it is evident that these arbitrary functions must also destroy each other, therefore

$$\psi x = f \left\{ x, \phi x, \underset{1}{\phi} x, \underset{2}{\phi} x, \text{\&c.} \right\}$$

is a general solution containing as many arbitrary functions as the particular solution did constant quantities

Examples, let

$$x\psi x = \psi \frac{1}{x}$$

a particular solution is $\psi x = a\frac{x+1}{x}$

and $\bar{\varphi}\left\{x, \frac{1}{x}\right\}$ does not change when x becomes $\frac{1}{x}$, therefore the general solution is

$$\psi x = \frac{x+1}{x}\bar{\varphi}\left(x, \frac{1}{x}\right)$$

take $$\psi x - \psi\frac{1}{x} = a\,(1 - x^2)$$

a particular case of this equation is

$$\psi x = a + bx$$

therefore, the general solution is

$$\psi x = a + \bar{\varphi}\left\{x, \frac{1}{x}\right\}.$$

Let $\psi\,(x) \times \psi\,(-x) = \frac{1+x}{1-x}\left(\frac{1}{x}\psi\frac{1}{x}\right)^2$

as a particular solution take

$$\psi x = \frac{1-x}{ax}$$

and $\varphi\left\{\frac{1-ax^2+x^4}{1+ax^2+x^4}\right\}$ is a functional equation which does not change when x becomes $-x$ or $\frac{1}{x}$, therefore the general solution is

$$\psi x = \frac{1-x}{x\varphi\left\{\frac{1-ax^2+x^4}{1+ax^2+x^4}\right\}}$$

φ being arbitrary.

Given the equation

$$2\psi\left(\frac{x}{\sqrt{x^2-1}}\right) = \psi x + \psi\,(-x) + \frac{\psi x - \psi\,(-x)}{\sqrt{x^2-1}}$$

a particular solution is

$$\psi x = a + bx$$

hence the general solution will be found to be

$$\psi x = \varphi\left(\frac{x^4}{x^2-1}\right) + x\underset{1}{\varphi}\left(\frac{x^4}{x^2-1}\right)$$

On the number of arbitrary functions introduced into the complete solution of a functional equation.

When from a functional equation of the first order, we determine the form of the unknown function, one or more constant quantities are generally introduced; these as I have shown in a preceding Problem, may be changed into arbitrary functions of the unknown quantity which fulfil certain prescribed conditions.

A question naturally arises as to the number of these arbitrary functions, and how many any given equation admits of in its most general solution.

The train of reasoning usually made use of to prove, that a differential equation of the n^{th} order, requires in its complete integral n, arbitrary constants may be pursued on the present occasion, though from several reasons, it would perhaps be desirable to have a proof resting on a different principle; as I have not been successful in discovering any other, I shall give the only one I am at present possessed of.

Let $$\psi\, x = F\left\{ x, \underset{1}{a}, \underset{2}{a}, \ldots \underset{n}{a} \right\}$$

for x, put any number of known functions, as αx, βx, .. νx, the results will be

$$\psi\, x = F\left\{ x, \underset{1}{a}, \underset{2}{a}, \ldots \underset{n}{a} \right\} \qquad (0)$$

$$\psi\, \alpha\, x = F\left\{ \alpha x, \underset{1}{a}, \underset{2}{a}, \ldots \underset{n}{a} \right\} \qquad (1)$$

$$\psi\, \beta\, x = F\left\{ \beta x, \underset{1}{a}, \underset{2}{a}, \ldots \underset{n}{a} \right\} \qquad (2)$$

&c. &c.

$$\psi\, \nu\, x = F\left\{ \nu x, \underset{1}{a}, \underset{2}{a}, \ldots \underset{n}{a} \right\} \qquad (n)$$

From these $n+1$ equations we may eliminate the n arbitrary constants, and the resulting equation will be of the form

$$o = F_1 \left\{ x, \psi x, \psi \alpha x, \ldots \psi \nu x \right\} \qquad (A)$$

In arriving at this equation, we have eliminated n arbitrary constants, and therefore it might possibly be inferred that the general solution of (A) is

$$\psi x = F \left\{ x, a_1, a_2, \ldots a_n \right\}.$$

But this is too hasty a conclusion, for it is evident, that we should equally have arrived at equation (A), if each of the constant quantities in (o) had been changed into a function of x so constituted that it should not alter by the substitution of αx, βx, &c. νx.

It would now appear, that putting such values for the constant quantities, the result would be the general solution of (A).

This reasoning is certainly plausible, and such a solution is undoubtedly a very general one; still, however, there are reasons which incline me to believe, that other solutions exist of a yet more general nature.

On functional equations of the second and higher orders.

When we consider functional equations of an order superior to the first, new difficulties present themselves; the artifices which were used with success in the preceding part of this paper, are no longer of any avail.

Those which we have now to consider seem to possess an entirely distinct character.

PROBLEM IX.

Required the solution of the equation

$$\psi^2 x = x \qquad (a)$$

Subtract ψx from both sides, then we have

$$\psi^2 x - \psi x = -\psi x + x = -(\psi x - x)$$

consequently,

$$\Delta \psi x = -\Delta x \qquad (b)$$

again multiply (a) by ψx, then we have

$$\psi^2 x \times \psi x = \psi x \times x$$

From this we learn, that (b) must be integrated on the hypothesis of $x \psi x$ being constant, hence

$$\psi x = -x + c = -x + f(x \psi x)$$

f being an arbitrary function

ψ is determined from the equation

$$\psi x + x - f(x \psi x) = o.$$

For f we may put $f^{-1} f_{\mathrm{I}}$, and it becomes

$$f^{\mathrm{I}}(x + \psi x) - f_{\mathrm{I}}(x \psi x) = o \qquad (c)$$

A friend to whose valuable remarks on this subject, I am much indebted, has communicated to me the following method of obtaining the same solution. Assume any symmetrical function of x and v.

$$\bar{\varphi}(x, v) = o$$

then from the nature of the equation we have the two following equations $\quad x = \psi v$ and $v = \psi x$

consequently

$$x = \psi v = \psi \psi x = \psi^2 x,$$

hence ψx may be found from the equation

$$\bar{\varphi}\left\{x, \psi x\right\} = o$$

This at first sight appears different from (c), but it is not so

in reality, for it may easily be shown, that from the sum and product of two quantities any symmetrical function may be composed.

As particular cases of the equation

$$\psi^2 x = x$$

we may notice

$$\psi x = a - x, \qquad \frac{x}{ax-1}, \qquad \frac{b-x}{1-ax}, \text{ and } \frac{ax + \sqrt{c + a^2 - 4x^2}}{2}$$

PROBLEM X.

Required the solution of

$$\psi^2 x = x$$

another solution may be found from the following principle.

Assume $$\psi x = \overset{-1}{\phi} f \phi x$$

hence $$\psi^2 x = \overset{-1}{\phi} f \phi \overset{-1}{\phi} f \phi x = \overset{-1}{\phi} f^2 \phi x$$

or $$\overset{-1}{\phi} f^2 \phi x = x.$$

Now this equation can be satisfied if we are acquainted with a particular solution of $f^2 x = x$, this may be found from the last problem, and since $f^2 x = x$, the equation becomes

$$\overset{-1}{\phi} \phi x = x$$

which is identical, consequently

$$\psi x = \overset{-1}{\phi} f \phi x$$

some particular cases are

$$\psi x = \overset{-1}{\phi} (a - \phi x,), \qquad \overset{-1}{\phi} \left(\frac{\phi x}{a \phi x - 1}\right), \qquad \overset{-1}{\phi} \left(\frac{a}{\phi x}\right) \text{ \&c.}$$

from each of these by assigning particular values to ϕ, new values of f may be determined, and these in their turn will furnish new forms of the function ψx.

Some time ago I received from the gentleman already alluded to, the following solution of the equation

$$\psi^2 x = x$$

$$\psi x = \phi (- \phi x)$$

and likewise the solution $\overset{-1}{\phi}\left((-1)^{\frac{1}{n}}\phi x\right)$ of the equation $\psi^n x = x$; this first led me to the substitution of $\overset{-1}{\phi} f \phi x$; which is of such essential use in these enquiries.

PROBLEM XI.

Given the equation.

$$\psi^n x = x.$$

Assume as before $\quad \psi x = \overset{-1}{\phi} f \phi x,$

then $$\psi^2 x = \overset{-1}{\phi} f \phi \overset{-1}{\phi} f \phi x = \overset{-1}{\phi} f^2 \phi x$$

$$\psi^3 x = \overset{-1}{\phi} f^2 \phi \overset{-1}{\phi} f \phi x = \overset{-1}{\phi} f^3 \phi x,$$

and generally $\quad \psi^n x = \overset{-1}{\phi} f^n \phi x,$

hence our equation becomes

$$\overset{-1}{\phi} f^n \phi x = x. \qquad (a)$$

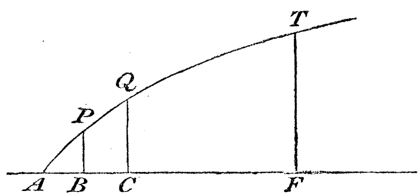
Suppose we have one particular solution of the equation, substitute this instead of f, and the equation (a) becomes identical, whence

$$\psi x = \overset{-1}{\phi} f \phi x$$

and from this other values of f may be determined, and so on ad infinitum.

The equation we have just considered, affords a ready solution of the following Problem.

Required the nature of a curve, such that taking any point B in the abscissa, and drawing the ordinate BP



if we make AC another abscissa equal to BP the preceding ordinate, and if we continue this n times, then the n^{th} ordinate may be equal to the first abscissa.

If AB $= x$ and the equation of the curve be $y = \psi x$.

PB $= y = \psi x$ and AC $=$ PB $= \psi x$,

and QC $= \psi \psi x = \psi^2 x$, and generally the n^{th} ordinate TF is equal to $\psi^n x$, hence

$$\psi^n x = x$$

which is the equation whose solution has been just found.

Problem XII.

Given the equation

$$\psi^2 x = \alpha x$$

required the form of ψ.

Assume $\qquad \psi x = \overset{-1}{\phi} f \phi x$

then $\qquad \psi^2 x = \overset{-1}{\phi} f^2 \phi x$

and $\qquad \overset{-1}{\phi} f^2 \phi x = \alpha x$

take the function ϕ on both sides, then this becomes

$$f^2 \phi x = \phi \alpha x.$$

This is a functional equation of the first order relative to ϕ, and may be solved either by the methods exhibited in the first part of this paper, or by the very elegant one of Laplace, f is a perfectly arbitrary function, except that neither fx nor $f^2 x$ must be equal to x: from not attending to this circumstance, I was at first led into several errors; the reason of these two restrictions is, that in the first case we at once determine ψx to be equal to x, and in the second, we in fact make $\alpha x = x$, neither of which are necessarily true.

Problem XIII.

Given the equation.

$$\psi^n x = \alpha x$$

This admits of a solution similar to the last, by assuming ψx

equal to $\overset{-1}{\phi} f \phi x$, we find

$$\overset{-1}{\phi} f^n \phi x = \alpha x$$

and by taking ϕ on both sides, it becomes

$$f^n \phi x = \phi \alpha x$$

give f some particular value and determine ϕ as in the last problem.

PROBLEM XIV.

Given the equation

$$\psi \alpha \psi \beta x = \gamma x$$

Assume $$\psi x = \overset{-1}{\phi} f \phi \underset{1}{f} x$$

hence $$\psi \alpha \psi \beta x = \overset{-1}{\phi} f \phi \underset{1}{f} \alpha \overset{-1}{\phi} f \phi \underset{1}{f} \beta x$$

make $\underset{1}{f} = \overset{-1}{\alpha}$, then the equation becomes

$$\overset{-1}{\phi} f \phi \overset{-1}{\phi} f \phi \overset{-1}{\alpha} \beta x = \overset{-1}{\phi} f^2 \phi \overset{-1}{\alpha} \beta x = \gamma x$$

and by taking ϕ on both sides

$$f^2 \phi \overset{-1}{\alpha} \beta x = \phi \gamma x$$

this is an equation of the first order relative to ϕ.

The equation $\psi \alpha \psi \alpha \,..\, (n) \, \psi \beta y = \gamma y$ might be solved in the same manner, but they are both reducible by a simple transformation to the form

$$\psi^n y = \mathrm{F} y$$

PROBLEM XIV.

Another method of solving the equation

$$\psi^2 x = x$$

Assume $\psi x = \overset{-1}{\phi} f \phi x$, then $\psi^2 x = \overset{-1}{\phi} f^2 \phi x$, and we have

$$\overset{-1}{\phi} f^2 \phi x = x.$$

Let $y = \phi x$, then $x = \overset{-1}{\phi} y$ and the equation becomes

$$\overset{-1}{\phi} f^2 y = \overset{-1}{\phi} y$$

an equation of the first order from which y may be found or thus assume

$$\overset{-1}{\phi} y = \bar{\chi}\left\{y, f^2 y\right\}$$

and make $f^4 y = y$,

this method is much more extensive in its application than any of those before it.

Problem XVI.

Required the solution of

$$\psi^n x = x$$

the same method applies equally in this case,

Assume $\psi x = \overset{-1}{\phi} f \phi x$

then $\overset{-1}{\phi} f^n \phi x = x$

for x put $\overset{-1}{\phi} x$ it becomes

$$\overset{-1}{\phi} f^n x = \overset{-1}{\phi} x.$$

Take $\overset{-1}{\phi}$ an arbitrary symmetrical function of

$$x, f^n x, f^{2n} x, \text{ and } f^{kn} x,$$

then $\overset{-1}{\phi} x = \chi\left\{x, f^n x, f^{2n} x, \ldots f^{kn} x\right\}$

and determine f such that

$$f^{(k+1)n} x = x$$

a particular solution is sufficient, and it is evident, this value of $\overset{-1}{\phi}$ will satisfy the equation.

Problem XVII.

Given the equation

$$\psi \alpha (y, \psi y) = y$$

required the form of y.

Assume $\psi y = \overset{-1}{\phi} f \phi y$, then it becomes

$$\overset{-1}{\phi} f \phi \alpha (y, \overset{-1}{\phi} f \phi y) = y,$$

take successively on each side the functions ϕ, f^{-1}, and $\overset{-1}{\phi}$, the equation becomes

$$\alpha\left(y, \overset{-1}{\phi} f \phi y\right) = \overset{-1}{\phi} f^{-1} \phi y$$

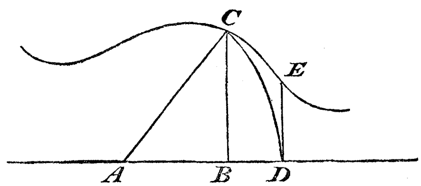
for y put $\overset{-1}{\phi} y$, then

$$\alpha\left(\overset{-1}{\phi} y, \overset{-1}{\phi} f y\right) = \overset{-1}{\phi} f^{-1} y,$$

this is a functional equation of the first order relative to $\overset{-1}{\phi}$, give f any determinate value and solve the equation.

From hence we may deduce the solution of the following question.

Required the nature of a curve



such that taking any abscissa AB and drawing the ordinate CB, if with centre A and radius AC, we describe a circle cutting the abscissa in D, the ordinate ED may be equal to the first abscissa AB.

Let $y = \psi x =$ CB

$$\mathrm{AD} = \mathrm{CA} = \sqrt{x^2 + (\psi x)^2}$$

and $\mathrm{ED} = \psi(\mathrm{AD})$, hence

$$x = \psi\left(\sqrt{x^2 + (\psi x)^2}\right)$$

which is a particular case of the preceding problem.

PROBLEM XVIII.

To reduce the equation

$$\mathrm{F}\left(x, \psi x, \psi^2 \alpha x, \ldots \psi^{n\nu} x\right) = 0$$

to one of the form of

$$F\left(x, \psi x, \psi^2 x, \ldots \psi^n x\right) = o.$$

Assume $\psi x = \overset{-1}{A} \phi A x$, the equation by this substitution becomes

$$F\left\{x, \overset{-1}{A} \phi A x, \overset{-1}{A} \phi^2 A \alpha x, \ldots \overset{-1}{A} \phi^n A \nu x\right\} = o$$

find by Problem VI. such a value Ax that it shall not change by the substitution of αx, βx, γx, &c. νx, put for Ax the quantity y, and the equation becomes

$$F\left\{\overset{-1}{A} y, \overset{-1}{A} \phi y, \overset{-1}{A} \phi^2 y, \ldots \overset{-1}{A} \phi^n y\right\} = o$$

which is an equation of the required form.

Problem XIX.

Required the solution of the equation

$$F\left\{x, \psi x, \psi^2 x, \ldots \psi^n x\right\} = o$$

Assume $\psi x = \overset{-1}{\phi} f \phi x$, then $\psi^n x = \overset{-1}{\phi} f^n \phi x$, and the equation becomes

$$F\left\{x, \overset{-1}{\phi} f \phi x, \overset{-1}{\phi} f^2 \phi x, \ldots \overset{-1}{\phi} f^n \phi x\right\} = o$$

for x substitute $\overset{-1}{\phi} x$, then

$$F\left\{\overset{-1}{\phi} x, \overset{-1}{\phi} f x, \overset{-1}{\phi} f^2 x, \ldots \phi^1 f^2 x\right\} = o \qquad (a)$$

which is an equation of the first order relative to $\overset{-1}{\phi}$ and may be solved by the methods in the beginning of this Paper, or, by means of the method given by Mr. Herschel, to which we have already alluded.

With respect to the function f it is arbitrary, there are however, some observations respecting it, which require notice; as without an attention to them we might fall into error. In the first place, it is evident, that we must not

make fx equal to x, for in this case we at once determine ψx to be equal to x, which is not always true.

The same observation may be made with respect to making $f^2 x = x$, for in this case $\psi^2 x = x$, $\psi^3 x = \psi x$, $\psi^4 x = x$, $\psi^5 x = \psi x$, and we in fact by assuming this value for f determine ψ from the equation

$$F\left\{x, \psi x, x, \psi x, x, \&c.\right\} = o$$

The same objection does not hold when we make $f^3 x = x$, though this considerably limits the generality of the solution; apparently the most eligible mode of determining f is from the equation $f^{n+1} x = x$, for in this case supposing we are acquainted with a particular solution of (a) containing any number of arbitrary constants, such as

$$\overset{-1}{\varphi} x = A\left\{x, a, b, c, \&c.\right\}$$

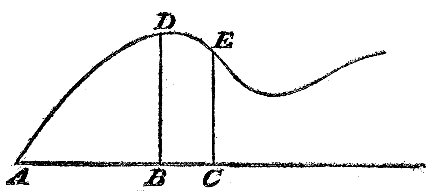
we may substitute for each of these constants an arbitrary function, such as

$$\bar{\chi}\left\{x, fx, f^2 x, \ldots f^n x\right\}$$

for it is evident this function does not change when we substitute $f^m x$ for x.

But this form of f is not always correct, it may be inconsistent with the original equation. In fact, the only value we can assign to f which shall not in some measure limit the generality of the Problem, is to suppose a particular solution of the given equation.

As an example of this method take the following problem.



Required the nature of the curve ADE such that taking any abscissa AB and corresponding ordinate DB: if the abscissa AC be taken, equal to DB and the ordinate EC be drawn, then the rectangle under the two ordinates shall be equal to the square of the first abscissa let the equation of the curve be $y = \psi x$ then $AB = x$ $DB = y = \psi x$ $AC = DB$,

and $EC = \psi\,(AC) = \psi\,(DB) = \psi y = \psi\psi x$

the given condition is therefore

$$\psi^2 x \times \psi x = x^2$$

making the usual substitution of $\psi x = \phi^{-1} f \phi x$ it becomes

$$\phi^{-1} f^2 \phi x \times \phi^{-1} f \phi x = x^2$$

putting $\phi^{-1} x$ for x we have

$$\phi^{-1} f^2 x \times \phi^{-1} f x = (\phi^{-1} x)^2$$

Assume $\phi^{-1} x = \bar{\chi}\,(x, fx, f^2x)$

the equation then becomes

$$\bar{\chi}\left\{f^2x, f^3x, f^4x\right\} \times \bar{\chi}\left\{fx, fx^2, fx^3\right\} = \left[\bar{\chi}\left\{x, fx, f^2x\right\}\right]^2$$

If now f be determined from the equation

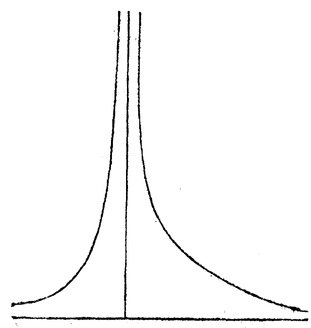
$$f^3 x = x$$

we have $f^4 x = fx$, and the equation becomes identical, hence ϕ^{-1} and f being found, we may determine ψ.

One of the simplest curves which satisfies the equation is

$$y\,x^2 = a^3$$

a line of the third order, the 69th in Newton's arrangement, its form is



PROBLEM XX.

Required the general solution of the equation

$$F\left\{x, \psi x, \psi^2 \alpha x, \psi^3 \beta x, \ldots \psi^n \nu x\right\} = 0$$

Reduce this by Problem XVIII. to the form of

$$F\left\{x, \psi x, \psi^2 x, \ldots \psi^n x\right\} = 0$$

in which shape it may be solved by the preceding problem.

PROBLEM XXI.

Given the equation

$$F\left\{x, \psi x, \psi\alpha\, (x, \psi \beta x)\right\} = 0$$

required its solution.

Assume $\qquad \alpha\,(x, \psi x) = \phi^{-1} f \phi x \qquad (a)$

hence $\qquad \psi x = \alpha^{1,-1}\,(x, \phi^{-1} f \phi x) \qquad (b)$

In the left side of this equation, put for x the left side of (a) and in the right side, put instead of x the right side of (a) the result is

$$\psi\alpha\,(x, \psi x) = \alpha^{1,-1}\,(\phi^{-1} f \phi x, \phi^{-1} f^2 \phi x)$$

substituting this in the original equation, the result is

$$F\left\{x, \alpha^{1,-1}\,(x, \phi^{-1} f \phi x), \alpha^{1,-1}(\phi^{-1} f \phi x, \phi^{-1} f^2 \phi x,)\right\} = 0$$

which by putting $\phi^{-1} x$ for x becomes

$$F\left\{\phi^{-1} x, \alpha^{1,-1}\,(\phi^{-1} x, \phi^{-1} f x), \alpha^{1,-1,}(\phi^{-1} f x, \phi^{-1} f^2 x)\right\} = 0$$

this being an equation of the first order relative to ϕ^{-1} may be solved as above.

With respect to the number of arbitrary functions which enter into the complete solution of functional equations of higher orders than the first, I have little at present to offer; the difficulty of the subject, and the wide extent of the enquiries

to which it would lead, induce me to postpone it until I have more time for the consideration. The following remarks may suffice for the present to point out some of its difficulties and the mode of enquiry.

If
$$\psi x = \underset{1}{f}\left\{x, a, b, \&c.\right\} \qquad (1)$$
$$\left.\begin{array}{l}\psi^2 x = \underset{2}{f}\left\{x, a, b, \&c.\right. \\ \psi^3 x = \underset{3}{f}\left\{x, a, b, \&c.\right.\end{array}\right\} \qquad \begin{array}{r}(2)\\(3)\end{array}$$
$$\&c. \qquad \&c.$$
$$\psi^n x = \underset{n}{f}\left\{x, a, b, \&c.\right\} \qquad (n)$$

From this by eliminating $n-1$ of the arbitrary constants a, b, &c. we have an equation of the form
$$F\left\{x, \psi x, \psi^2 x, \ldots \psi^n x\right\} = o \qquad (a)$$
and it might possibly be concluded that equation (1) containing $n-1$ arbitrary constants is the general solution of this last equation: but this is by no means the case. In the first place between the two equations (1) and (2), more than one arbitrary constant may be eliminated, thus let
$$\psi x = \frac{a-x}{1-bx}$$
from which we find
$$\psi^2 x = x$$
the two quantities a and b have been eliminated, and it is possible to select a value of ψx, between which and $\psi^2 x$ an infinite number of arbitrary constants could be eliminated.

But waving this objection let us consider the case of (a) which is deduced from the elimination of $n-1$ arbitrary functions.

We have seen in Problem VI. that a function of the first

order may satisfy any number of conditions (which are not contradictory) simultaneously; and there appears no reason for denying this property to those of higher orders.

If now we consider the symmetrical function

$$\bar{\chi}\left\{x, \psi x, \psi^2 x, \ldots \psi^v x\right\}$$

and if $\psi^{v+1} x = x$. It is evident this function will not change by the substitution of ψx, $\psi^2 x$, &c. or $\psi^n x$
and consequently that a different function similarly constituted may be substituted for each of the arbitrary quantities a, b, c, &c. in (1) which is the solution of the equation

$$\mathrm{F}\left\{x, \psi x, \psi^2 x, \ldots \psi^n x\right\} = 0 \qquad (a)$$

The number denoted by v is arbitrary (it may, however, become determined from some particular circumstances of the equation (a).

Thus we have introduced into an equation of the n^{th} order, an unlimited number of arbitrary functions, each of which contains the function whose determination was sought with all its different orders to an undefined extent.

If we take the particular case of

$$\psi x = a - x$$
$$\psi^2 x = x$$

v must be unity, and a general solution is

$$\psi x = \bar{\chi}\left\{x, \psi x\right\} - x$$

taking another solution

$$\psi x = \frac{a-x}{1-bx}, \qquad \psi^2 x = x$$

and

$$\psi x = \frac{\bar{\chi}(x, \psi x) - x}{1 - x\,\bar{\chi}(x, \psi x)}$$

If $\psi^n x = x$ $\qquad v = n - 1$, and supposing $\psi x = f(x, a, b, \&c.)$ any particular solution we have for the general one

$$\psi x = f\left\{x, \bar{\chi}(x, \psi x, \ldots \psi^{n-1} x), \bar{\chi}_1(x, \psi x, \ldots \psi^{n-1} x) \text{ \&c.}\right\}$$

from which equation of the $\overline{n-1}^{\text{th}}$ degree ψ must be found.

When we apply these considerations to functional equations of many variables, other and even greater difficulties present themselves; the first step in that direction must be an improvement in the notation.

Since the above was written, I have bestowed some attention on functional equations involving two or more variables, and I have met with considerable success: I am in possession of methods which give the general solution of equations of all orders, and even of those which contain symmetrical functions. I have also discovered a new and direct method of treating functional equations of the first order, and of any number of variables, and this new method I have applied to the solution of differential and even of partial differential functional equations.